Nonabelian Cohomology

WANG Jinyi

version 2025-10-19

1 Preliminaries

1.1 ∞-Categories and Anima

We will be speaking the language of ∞-categories which is based on the concept of *anima*.

An anima is like a set; it contains the information of a *collection* of mathematical objects, but it moreover contains the information of *equations* between them, and *equations between equations* (in fact, the *collection* of equations between two objects is again an anima), and so on. Thus an anima knows more than a set, which only remembers *equivalence classes* of mathematical objects.

The name *anima* is a synonym for ∞-*groupoid*, but has the emphasis that it should be the *most fundamental* concept of mathematics, rather than a derived one.

The traditional notion of *category* is based on sets. If we are to adopt the new concept of anima, then categories would become ∞-*categories*. The most important ∞-category is the category Ani of animas, which plays the role of Set in ordinary category theory.

The theory of ∞-categories uses essencially the same language as the theory of ordinary categories, but interpreted differently; the difference is exactly caused by the shift from set-based mathematics to anima-based mathematics: for example, when we say two things are *equal*, we must always remember a *specific equivalence* between them, because equations between every two objects x, y form an *anima*, which we denote by $x = y$.

Traditionally we were not allowed to say two structures are *equal*, only *equivalent*, because we did not carry the equivalence with us (that is, we often *truncate* the equailty $x = y$ into a *mere proposition*; see below). Now since we always remember equivalences, we can talk about equality of any structures just like we talk about equality between elements of a set. The most succesful example of this kind of language is *Homotopy type theory*.

Truncatedness, and the distinction Structures v.s. Properties.

A feature of the ∞-categorical language is that we carefully use certain terminologies borrowed from the natural language to indicate *truncatedness*.

- A (-2)-truncated anima (also called *contractible anima* or `true` in HoTT) is just equivalent to $*$.
- A (-1)-truncated anima (also called a *mere proposition* in HoTT) is either empty or contractible. Equivalently, a (-1)-truncated anima is one in which *every two objects are equal*; the equality $x = y$ between every two objects x, y is simply `true`.

- A 0-truncated anima (also called a *set*) is one in which every connected component is contractible. Equivalently, a set is one in which the equality $x = y$ between every two objects x, y is a mere proposition.
- A 1-truncated anima (also called a 1-*groupoid*) is one in which the equality $x = y$ between every two objects x, y is a set. A 1-groupoid can be presented by a set S of objects and a set of isomorphisms $\mathrm{Hom}(x, y)$ between $x, y \in S$. Note however that this is only a presentation and not the object itself.
- etc.

The recursive definition is (you guessed it)

- An n-truncated anima is one in which the *equations* between any two objects form an $(n-1)$-truncated anima.

Examples.

- In an ordinary group, a equation like $x + y = z$ is a property because "either they are equal or they are not"; they cannot be equal *in more than one way*.
- In an ordinary ring, that "x is invertible" is a property because the set of inverses of x is either empty or equivalent to $*$. An element cannot have more than one inverses.
- For any morphism $f : X \to Y$ in a category, that "f is invertible" is a property. This holds even in the ∞-categorical context!
- For two objects X, Y in a category, that "X and Y are equivalent" is *not* a property because the anima of equivalences between X and Y is generally not (-1)-truncated.
- In an ordinary category like Set, that a group is *commutative* is a property because morphisms form a set and thus equalities between morphisms form a (-1)-truncated anima; this is no longer true in higher categories: commutativity is an extra *structure*.

Truncatedness is kind of orthogonal to *connectedness*: an anima is n-truncated if its homotopy groups *above* degree n are trivial; it is n-connective if its homotopy groups *below* degree n are trivial (connected = 1-connective). If a pointed anima is both n-connective and n-truncated, it is called an *Eilenberg–Maclane space*.

An extra note about presentations.

Traditional homotopy theory, in the lack of suitable language, have used various 1-categorical presentations of the ∞-category Ani.

- Animas are presented by topological spaces or certain simplicial sets (Kan complexes).
- *(Co)limits* are presented by *homotopy (co)limits*.
- etc.

It should be noted that these presentations are often unnecessary; we can do many (if not all) computations without them using the correct language.

1.2 Toposes

The concept of a *topos* is a category that looks like Set in so many ways that whenever we "properly" say something about sets (that is, without using the axiom of choice or the law of excludded middle), we can say the same thing in any topos, in a manner that *pretends the objects of the category are sets.*

The most important examples of toposes are *categories of sheaves on sites.* Given any category $\mathcal{C}$ of *geometric objects*, e.g. manifolds, schemes, and a suitable sense of *covering*, we get a *site* and may form the category of sheaves on it. This category *completes* the site $\mathcal{C}$ in a precise way, and its objects can be thought of as generalized spaces *glued out of* (or *locally modeled on*) objects of $\mathcal{C}$.

However note that a site is only a *presentation* of a topos, just like an algebraic equation is a presentation of a ring, or a differential equation is a presentation of a D-module. Since it is only a presentation, there can be different sites presenting the same topos. For example the site $(\mathsf{Sch}, \mathrm{fppf})$ (schemes equipped with the fppf topology) presents the same topos as its subcategory $(\mathsf{Aff}, \mathrm{fppf})$ (affine schemes) with the induced topology. Or the site CHaus of compact Hausdorff spaces presents the same topos as its subcategory ProFin of profinite sets. This is an important feature of *condensed mathematics.*

In the context of anima-based mathematics, toposes become ∞-toposes. Therefore, it is not surprising that *what we can say properly about animas can also be said in any ∞-topos*, in a manner that *pretends the objects in the category are animas.* This enables us to apply almost all the essencial results in homotopy theory to any ∞-topos. (but beware that Whitehead's theorem is no longer a theorem but an axiom called *hypercompleteness* we may have to impose.)

As in the old theory, the category of Ani-valued sheaves on a site gives an ∞-topos. The ∞-topos generated by a site *completes* the site in a deeper way, and its objects can be thought of as even more generalized spaces that allow "points" to have nontrivial automorphisms. For example, the category of *orbifolds* embeds into the ∞-topos generated by the category of manifolds. Historically, people used fibered categories (fibered in groupoids) to describe groupoid-valued sheaves. They are equivalent to functors from $\mathcal{C}^{\mathrm{op}}$ to Ani, by a process called straightening.

There should be a remark about *petit toposes* and *gros toposes.* (todo)

1.3 Groups and Delooping

The concept of *groups* has a generalization in ∞-categorical contexts, also known as $\mathbb{A}_\infty$-*groups* or $\mathbb{E}_1$-*groups*, made precise by the theory of operads. Moreover, they have particularly nice descriptions in ∞-toposes.

A *group* in Ani, as well as in any ∞-topos, is equivalent to a *pointed connected object*; i.e. there is a pair of equivalence

$$\mathbf{B} : \mathsf{Grp}(\mathsf{Ani}) \rightleftarrows \mathsf{Ani}^{\geq 1}_{*/} : \Omega$$

that relates a group G with its *delooping* $\mathbf{B}G$, which is a pointed connected object. Here Ω is the loopspace functor

$$\Omega X = * \times_X * \,.$$

Put in another way, for any anima X (thought of as a *type* of things) with a point $x : * \to X$ (thought of as an *instance* of it), the loopspace $\Omega(X, x)$ is the equality $x = x$, i.e. the collection of *automorphisms* of x in X. This is exactly what a group is supposed to mean, philosophically.

An ordinary group G in Set of course gives a group in Ani: its delooping is the one-object 1-groupoid $\mathbf{B}G$ with set of morphisms G.

Sometimes a group G can be delooped more than once. If the delooping $\mathbf{B}G$ is further equipped with a group structure, then G has the structure of an $\mathbb{E}_2$*-group*. Equivalently, G can be obtained from taking the second loopspace $\Omega^2 X$ for some pointed *simply-connected* object $X = \mathbf{B}^2 G$.

Similarly, an $\mathbb{E}_n$*-group* is one that can be delooped n times in this manner. They are exactly obtained from taking n-fold loopspaces of n*-connective objects*: there is an equivalence

$$\mathbf{B}^n : \mathsf{Grp}_{\mathbb{E}_n}(\mathsf{Ani}) = \underbrace{\mathsf{Grp}(\mathsf{Grp}(\cdots\mathsf{Grp}(\mathsf{Ani})))}_{n \text{ times}} \rightleftarrows \mathsf{Ani}_{*/}^{\geq n} : \Omega^n.$$

One should keep in mind that a group being $\mathbb{E}_n$ is generally an extra *structure*.

A *commutative group* (also known as an $\mathbb{E}_\infty$-group) is one that can be delooped *arbitrarily* many times. Thus they are also called *infinity loop spaces*.

Example. In an ordinary category like Set, an $\mathbb{E}_2$-group is already a *commutative group*,

$$\mathsf{Grp}_{\mathbb{E}_2}(\mathsf{Set}) = \mathsf{Grp}(\mathsf{Grp}(\mathsf{Set})) = \mathsf{Ab},$$

and $\mathbb{E}_n$-groups with $n > 2$ give nothing more—because commutativity here is only a *property*. This is why homotopy groups π_n are all abelian when $n \geq 2$. (The pointed sphere (S^n, s_0) is an $\mathbb{E}_n$-cogroup in $\mathsf{Ani}_{*/}$, so the pointed mapping space $\Omega^n(X, x_0)$ is an $\mathbb{E}_n$-group in Ani and $\pi_n(X, x_0) = \pi_0 \Omega^n(X, x_0)$ is an $\mathbb{E}_n$-group in Set.)

2 Cohomology: a general idea

All notions of cohomology are special cases of the following idea: for two objects X, A in an ∞-category $\mathcal{C}$ (often an ∞-topos, e.g. the category $\mathsf{Sh}(X, \mathsf{Ani})$ of sheaves on a space), the degree-0 cohomology of X with coefficients in A is

$$H^0(X, A) := \pi_0 \operatorname{Hom}_{\mathcal{C}}(X, A).$$

If moreover A is equipped with an n-fold delooping $\mathbf{B}^n A$, then the degree-n cohomology of X with coefficients in A can be defined as

$$H^n(X, A) := H^0(X, \mathbf{B}^n A) = \pi_0 \operatorname{Hom}_{\mathcal{C}}(X, \mathbf{B}^n A).$$

Classically the object A is an abelian group (also known as an $\mathbb{E}_\infty$-group), and thus have arbitrary deloopings $\mathbf{B}^n A$. Special cases of this construction include:

- Ordinary cohomology. For a good topological space X and an abelian group A,

$$H^n(X, A) = \pi_0 \operatorname{Hom}_{\mathsf{Top}}(X, \mathbf{B}^n A),$$

where $\mathbf{B}^n A$ is known as the n-th Eilenberg–Maclane space of A.

- Sheaf cohomology. For a sheaf A of abelian groups on a space X,

$$H^n(X, A) = \pi_0 \operatorname{Hom}_{\mathsf{Sh}(X,\mathsf{Ani})}(\underline{1}, \mathbf{B}^n A) = \pi_0 \operatorname{Hom}_{\mathsf{Sh}(X,\mathsf{Ab}(\mathsf{Ani}))}(\underline{\mathbb{Z}}, \mathbf{B}^n A),$$

where $\underline{1}$ and $\underline{\mathbb{Z}}$ are constant sheaves on X.

- Group cohomology. For a group G acting on an abelian group A, the *group cohomology*

$$H^n_{\mathrm{Grp}}(G, A) = \pi_0 \operatorname{Hom}_{\mathsf{Ani}/\mathbf{B}G}(\mathbf{B}G, (\mathbf{B}^n A)/G) = \pi_0 \operatorname{Hom}_{\mathsf{Sh}(\mathbf{B}G,\mathsf{Ani})}(\underline{1}, \mathbf{B}^n(A/G))$$

is just the sheaf cohomology of the sheaf A/G of abelian groups on $\mathbf{B}G$. (The pullback functor or "fiber functor" $p^* : \mathsf{Sh}(\mathbf{B}G, \mathsf{Ani}) = \mathsf{Ani}_{/\mathbf{B}G} \to \mathsf{Ani}$ preserves finite limits and sends the sheaf A/G abelian groups to the abelian group A.) This is an example of *twisted cohomology*. For the trivial action of G on A, the group cohomology reduces to the ordinary cohomology of $\mathbf{B}G$:

$$H^n_{\mathrm{Grp}}(G, A_{\mathrm{triv}}) = \pi_0 \operatorname{Hom}_{\mathsf{Ani}/\mathbf{B}G}(\mathbf{B}G, \mathbf{B}^n A \times \mathbf{B}G) = \pi_0 \operatorname{Hom}_{\mathsf{Ani}}(\mathbf{B}G, \mathbf{B}^n A) = H^n(\mathbf{B}G, A).$$

3 Torsors and nonabelian H^1

In Ani as well as in any ∞-topos, for a group G there is an equivalence

$$G\text{-Act} := \{\text{object } X \text{ with } G\text{-action}\} \rightleftarrows \{\text{maps } Y \to \mathbf{B}G\} = \mathsf{Ani}_{/\mathbf{B}G}.$$

The correspondence is as follows:

- For an G-action X, there is a map $(X/G) \to (*\,/G) = \mathbf{B}G$.
- For a map $Y \to \mathbf{B}G$ we may form the pullback

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \mathbf{B}G \end{array} \qquad (\star)$$

where X carries a natural G-action. The G-bundle $X \to Y$ is then called a *G-torsor*, or traditionally *principal G-bundle*. Therefore we may say that equivalence classes of G-torsors on Y are classified by the first nonabelian cohomology

$$H^1(Y, G) = \pi_0 \operatorname{Hom}(Y, \mathbf{B}G).$$

3.1 More explicit structures of a torsor

One may wonder why the traditional concept of principal bundles is a special case of the above. This can be explained as follows.

Given a map $X \to Y$, the *Čech nerve* $\check{\mathrm{C}}(X \to Y)$ is the augmented simplicial object

$$\cdots \qquad X \times_Y X \times_Y X \rightrightarrows X \times_Y X \rightrightarrows X \longrightarrow Y.$$

An immediate consequence of the correspondence between groups G and their delooping $\mathbf{B}G$ is that the Čech nerve of the basepoint map $* \to \mathbf{B}G$ is

$$\cdots \qquad G \times G \rightrightarrows G \rightrightarrows * \longrightarrow \mathbf{B}G.$$

Forming Čech nerves is a kind of limit, and in Ani as well as in any ∞-topos, limits are compatible with pullbacks. Therefore from the pullback diagram ($\star$), we get a pullback diagram of simplicial objects

$$\begin{array}{ccccccccc}
\cdots & X \times_Y X \times_Y X & \rightrightarrows & X \times_Y X & \rightrightarrows & X & \longrightarrow & Y \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & G \times G & \rightrightarrows & G & \rightrightarrows & * & \longrightarrow & \mathbf{B}G
\end{array}$$

where every square is a pullback, and from which we deduce equivalences

$$X \times_Y X = G \times X, \qquad X \times_Y X \times_Y X = G \times G \times X, \qquad \text{etc.}$$

The first equivalence is exactly what we require for a traditional principal bundle, and the rest are coherence conditions that appear in higher categorical contexts.

Maybe to one's surprise, we have *local triviality* of torsors even in the ∞-categorical context. This is made precise as follows. A bundle $X \to Y$ is called *locally trivial* if for some epimorphism $U \to Y$ we have

$$X \times_Y U = F \times U$$

for some object F. (Such a map $X \to Y$ is called an F-bundle.) If $X \to Y$ is a G-torsor, then it is also an epimorphism and it trivializes itself: $X \times_Y X = G \times X$.

3.2 Examples: vector bundles on stacks

This section describes an important class of examples of torsors classified by H^1.

Let Sch be the category of schemes, or S-schemes for any base scheme S if you like, equipped with the fppf or étale topology. Consider the ∞-topos Stk of Ani-valued sheaves on the site Sch. Objects of this topos are called *stacks*. Algebraic stacks are special stacks satisfying certain representability conditions. Of course, *schemes* themselves are (0-truncated) objects of Stk by Yoneda embedding.

The group scheme $\mathrm{GL}_n : \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Grp}$ is a 0-truncated group in the ∞-topos Stk. Its delooping $\mathbf{B}\mathrm{GL}_n$ is a 1-truncated pointed connected object of Stk and, by the general theory, classifies GL_n-torsors on stacks.

It is a well-known fact in algebraic geometry that GL_n-torsors on a scheme are equivalent to *rank-n vector bundles* on it.

For any stack X, define $\mathrm{Bun}_n(X)$ to be the internal hom $[X, \mathbf{B}\mathrm{GL}_n]$, which is a 1-truncated object of Stk. The anima of points of $\mathrm{Bun}_n(X)$, i.e. the anima $\mathrm{Hom}(X, \mathbf{B}\mathrm{GL}_n)$, is the (1 -)groupoid of rank-n bundles over X. The set of connected components

$$H^1(X, \mathrm{GL}_n) = \pi_0 \operatorname{Hom}(X, \mathbf{B}\mathrm{GL}_n)$$

is then the set of equivalence classes of rank-n bundles on X.

In particular, the *Picard stack*

$$\mathrm{Pic}(X) := \mathrm{Bun}_1(X) = [X, \mathbf{B}\mathbb{G}_m]$$

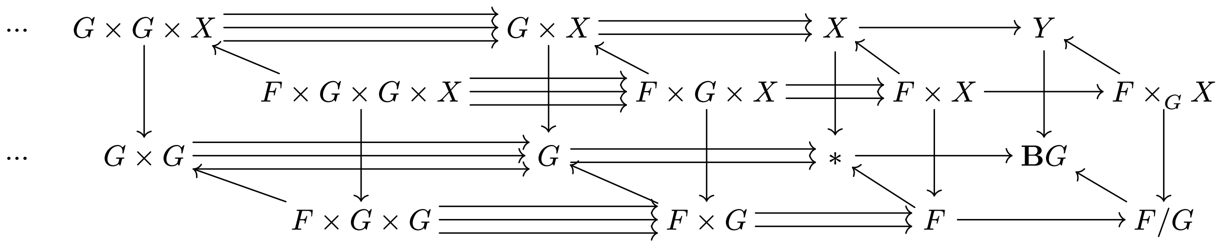
classifies *line bundles* on X. The set of components $H^1(X, \mathbb{G}_m) = \pi_0 \operatorname{Hom}(X, \mathbf{B}\mathbb{G}_m)$ is the set of equivalence classes of line bundles on X.

4 Associated bundles and twisted cohomology

Given an action of a group G on an object F, as a map $(F/G) \to \mathbf{B}G$, and a G-torsor $X \to Y$ as a map $Y \to \mathbf{B}G$, we may form the pullback

$$\begin{array}{ccc} F \times_G X & \longrightarrow & F/G \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \mathbf{B}G \end{array}$$

to get an F-bundle $F \times_G X \to Y$. The object $F \times_G X$ is also a quotient $(F \times X)/G$, as seen in the following pullback diagram (recall that pullbacks preserve colimits in any ∞-topos, and a quotient is the colimit of a cosimplical diagram).



The anima of sections of the associated bundle are equivalent to morphisms in the slice topos:

$$\Gamma(Y, F \times_G X) = \operatorname{Hom}_{\mathsf{Ani}/\mathbf{B}G}(Y, F/G).$$

The set of components

$$H^X(Y, F) := \pi_0 \operatorname{Hom}_{\mathsf{Ani}/\mathbf{B}G}(Y, F/G)$$

is sometimes called the X-*twisted cohomology* of Y with coefficients in F.

Example. Often the group G acting on the fiber F is just taken to be the automorphism group $\operatorname{Aut}(F)$. In fact, any F-bundle is canonically associated to an $\operatorname{Aut}(F)$-torsor.

Example. If the object F is pointed and connected, so that $F = \mathbf{B}A$ for some group A, then we may denote the “first twisted cohomology” of Y by

$$H^{1,X}(Y, A) := \pi_0 \operatorname{Hom}_{\mathsf{Ani}/\mathbf{B}G}(Y, \mathbf{B}A/G).$$

If F is pointed and n-connective, so that $F = \mathbf{B}^n A$ for some $\mathbb{E}_n$-group A, then we may denote the “n-th twisted cohomology” of Y by

$$H^{n,X}(Y, A) := \pi_0 \operatorname{Hom}_{\mathsf{Ani}/\mathbf{B}G}(Y, \mathbf{B}^n A/G).$$

Of course, these definitions are reduced to ordinary (sheaf) cohomology when X is the trivial F-bundle.

5 Short exact sequences and fiber sequences

For $G' \to G \to G''$ a short exact sequence of groups, we have the following sequence of pullback squares,

$$\begin{array}{ccccccc}
G' & \longrightarrow & G & \longrightarrow & * & & \\
\downarrow & & \downarrow & & \downarrow & & \\
* & \longrightarrow & G'' & \longrightarrow & \mathbf{B}G' & \longrightarrow & * \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & * & \longrightarrow & \mathbf{B}G & \longrightarrow & \mathbf{B}G''
\end{array}$$

In other words, we have a fiber sequence

$$G' \to G \to G'' \to \mathbf{B}G' \to \mathbf{B}G \to \mathbf{B}G''$$

One may take this as the *definition* of short exact sequences (of groups in a topos).

A fiber sequence like the above gives an exact sequence

$$H^0(X, G') \to H^0(X, G) \to H^0(X, G'') \to H^1(X, G') \to H^1(X, G) \to H^1(X, G'')$$

for any X. Without further commutativity structures, this is all we have. Now we have two things to do:

1. we can *require more* about the groups G', G, G'' in order to extend the fiber sequence to the right:
$$\cdots \to \mathbf{B}G'' \to \mathbf{B}^2 G' \to ??\,.$$
2. we can *approximate* given groups G', G, G'' using better structures that fit the requirements above.

Example. If $A \to \hat{G} \to G$ is a central extension of ordinary (0-truncated) groups, then there is a fiber sequence

$$A \to \hat{G} \to G \to \mathbf{B}A \to \mathbf{B}\hat{G} \to \mathbf{B}G \to \mathbf{B}^2 A.$$

In fact, central extensions of G by A are classified by homotopy classes of maps $\mathbf{B}G \to \mathbf{B}^2 A$, i.e. $H^2_{\mathrm{Grp}}(G, A)$.

6 Higher nonabelian cohomology

6.1 Gerbes

A *gerbe* (a French name whose meaning is close to “sheaf”) inside an ∞-topos is simply a connected object.

Remark. Giraud’s original definition of a gerbe, written in the cumbersome language of fibered categories on sites, is essencially a 1-truncated connected object (in the ∞-topos presented by a site).

If a gerbe X is moreover *pointed*, i.e. equipped with a map $x_0 : * \to X$, then it has the form $\mathbf{B}G$ for some group $G = \Omega(X, x_0)$.

Note that X being connected requires $X \to *$ to be an epimorphism, but this *does not* imply the existence of a *global point* $* \to X$. For example a G-torsor $X \to *$ in a topos is an epimorphism, but only a *trivial* torsor can have a global point. This is the failure of the *axiom of choice* in general toposes.

Although gerbes are not necessarily of the form $\mathbf{B}G$, they may be *locally* so. Given a group G, a G-gerbe is a gerbe X with an epimorphism $U \to *$ and an equivalence

$$X \times U = \mathbf{B}G \times U.$$

In other words, a G-gerbe is simply a $\mathbf{B}G$-bundle over $*$. Therefore it is canonically associated to an $\operatorname{Aut}(\mathbf{B}G)$-torsor and is classified by $H^1(*, \operatorname{Aut}(\mathbf{B}G))$.

Example. When G is an ordinary group (i.e. an 0-truncated group in Ani), $\operatorname{Aut}(\mathbf{B}G)$ is the quotient $\operatorname{Aut}_{\mathsf{Grp}}(G)/_{\mathrm{Ad}}G$ of the set $\operatorname{Aut}_{\mathsf{Grp}}(G)$ by the adjoint action of G. This is related to the fact that

$$\operatorname{Hom}(\mathbf{B}H, \mathbf{B}G) = \operatorname{Hom}_{\mathsf{Grp}}(H, G)/_{\mathrm{Ad}}G.$$

Note that $\operatorname{Hom}_{\mathsf{Grp}}(H, G) = \operatorname{Hom}_{\mathsf{Ani}_{*/}}(\mathbf{B}H, \mathbf{B}G)$ and there is a natural map $\operatorname{Hom}_{\mathsf{Ani}_{*/}}(\mathbf{B}H, \mathbf{B}G) \to \operatorname{Hom}_{\mathsf{Ani}}(\mathbf{B}H, \mathbf{B}G)$ forgetting the point, which turns out to be a quotient map by a G-action.

6.2 Outer automorphisms

Let G be a 0-truncated group. The *outer automorphism group* $\operatorname{Out}(G)$ is defined by the truncation

$$\operatorname{Out}(G) = \tau_{\leq 0}\big(\operatorname{Aut}_{\mathsf{Grp}}(G)/_{\mathrm{Ad}}G\big) = \tau_{\leq 0} \operatorname{Aut}(\mathbf{B}G).$$

Note that $\mathbf{B}G$ and $\operatorname{Aut}(\mathbf{B}G)$ are 1-truncated. If we regard $\operatorname{Aut}(\mathbf{B}G)$ as a 1-groupoid, then $\operatorname{Out}(G)$ is its set of isomorphism classes. The fiber of $\operatorname{Aut}(\mathbf{B}G) \to \operatorname{Out}(G)$ at $\mathrm{id} : * \to \operatorname{Aut}(\mathbf{B}G)$ is the connected component of $\mathrm{id}_{\mathbf{B}G} \in \operatorname{Aut}(\mathbf{B}G)$ (also $\mathrm{id}_G \in \operatorname{Aut}_{\mathsf{Grp}}(G)/_{\mathrm{Ad}}G$), which is $\mathbf{B}\operatorname{Aut}_{\operatorname{Aut}(\mathbf{B}G)}(\mathrm{id}_{\mathbf{B}G})$ (also $\mathbf{B}\operatorname{Aut}_{\operatorname{Aut}_{\mathsf{Grp}}(G)/_{\mathrm{Ad}}G}(\mathrm{id}_G)$). In fact, the group

$$\operatorname{Aut}_{\operatorname{Aut}(\mathbf{B}G)}(\mathrm{id}_{\mathbf{B}G}) = \big\{g \in G \mid \mathrm{Ad}_g \mathrm{id}_G = \mathrm{id}_G\big\} = \big\{g \in G \mid \forall h \in G, ghg^{-1} = h\big\}$$

is the center $Z(G)$ of G. This gives a remarkable definition of center which apparently admits much generalizations.

Therefore, there is a fiber sequence

$$\mathbf{B}^2 Z(G) \to \mathbf{B}\operatorname{Aut}(\mathbf{B}G) \to \mathbf{B}\operatorname{Out}(G)$$

given an exact sequence

$$H^2(X, Z(G)) \to H^1(X, \operatorname{Aut}(\mathbf{B}G)) \to H^1(X, \operatorname{Out}(G))$$

Giraud called an element of $H^1(X, \operatorname{Out}(G))$ a *lien* (a French name meaning “bond, contact, link”).

6.3 Truncation, the Postnikov tower and its fibers

In Ani as well as in any ∞-topos, the inclusion of the subcategory of n-truncated objects $\mathsf{Ani}_{\leq n} \hookrightarrow \mathsf{Ani}$ admits a left adjoint $\tau_{\leq n}$ called *truncation*. Explicitly, the adjoint property states that for any n-truncated object Y, the map

$$\mathrm{Hom}(\tau_{\leq n}X, Y) \to \mathrm{Hom}(X, Y)$$

is an equivalence.

Moreover, the truncation $\tau_{\leq n} : \mathsf{Ani} \to \mathsf{Ani}_{\leq n}$ preserves products, and thus preserves $\mathbb{E}_k$-group structures.

Some low-level truncations in Ani:
- $\tau_{\leq -2}X$ is always $*$.
- $\tau_{\leq -1}X$ is either $\emptyset$ or $*$ depending on X being empty or nonempty.
- $\tau_{\leq 0}X$ is the set of components of X.

For an object X we may form the *Postnikov tower*

$$X \to \cdots \to \tau_{\leq 0}X \to \tau_{\leq -1}X \to \tau_{\leq -2}X = *,$$

Example. The Postnikov tower of S^2 is

$$S^2 \to \cdots \to \tau_{\leq 3}S^2 \to \mathbf{B}^2\mathbb{Z}(= \mathbb{C}P^\infty) \to * \to * \to * \to * \,.$$

For $n \geq 0$, the fiber of a Postnikov tower

$$\mathrm{fib}(\tau_{\leq n}X \to \tau_{\leq n-1}X)$$

is an Eilenberg–Maclane space $\mathbf{B}^n\pi_n(X)$. This gives a short exact sequence

$$H^n(*, \pi_n(X)) \to H^0(*, \tau_{\leq n}X) \to H^0(*, \tau_{\leq n-1}X).$$

This shows that general nonabelian cohomology is controlled by (a successive extension of) ordinary cohomology and torsor theory.