

∞ -GROUP THEORY

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ABSTRACT. This article is an exposition of group theory reformulated in the intrinsic language of ∞ -categories. We develop basic structures of group theory including centralizers, normalizers and Weyl groups, revealing categorical properties that are obscured in the set-theoretic formulation.

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1. INTRODUCTION

A group is the abstraction of the collection of automorphisms of an object in a category.

This article is an exposition that rewrites classical group theory in the intrinsic language of ∞ -categories. Our choice of language is made so that many definitions can be understood *verbatim* in both classical and ∞ -group theories, thus justifying our definitions and saving us some trouble of proving compatibility. To the author, it is very admirable that many concepts and facts which at first sight can only make sense in classical group theory actually appear naturally in the context of ∞ -group theory.

2. CONVENTIONS

This article uses intrinsic $(\infty, 1)$ -categorical language; a *category* is a $(\infty, 1)$ -category. Limits and colimits are in the $(\infty, 1)$ -categorical sense. An *anima* is an ∞ -groupoid, or equivalently a $(\infty, 0)$ -category. The category of anima is denoted \mathbf{Ani} .

We note that discussions in this article apply not only to \mathbf{Ani} but also to any $(\infty, 1)$ -topos, if interpreted correctly.

3. BASICS

3.1. Group.

Definition 3.1.1. A *group* is a \mathbb{E}_1 -algebra in \mathbf{Ani} in which every element is invertible. The category of groups is denoted \mathbf{Grp} . Morphisms between groups are called *group homomorphisms*.

The following characterization of groups is what we use throughout this article.

Proposition 3.1.2. A group is equivalently a pointed connected anima. Specifically there is a pair of equivalences

$$\mathbf{Ani}_{*/}^{\geq 1} \begin{array}{c} \xrightarrow{\Omega} \\ \xleftarrow{B} \end{array} \mathbf{Grp}.$$

Example 3.1.3 (automorphism group, loop group). For any category \mathcal{C} and any object $x \in \mathcal{C}$, we have the *automorphism group* $\mathrm{Aut}_{\mathcal{C}}(x) \in \mathbf{Grp}$ with the property that $B \mathrm{Aut}(x)$ is the connected component of \mathcal{C}^{\simeq} containing x . This construction can be upgraded into a functor

$$\mathrm{Aut}_{\mathcal{C}}(-): \mathcal{C}^{\simeq} \rightarrow \mathbf{Grp}.$$

We note that the underlying anima of $\mathrm{Aut}_{\mathcal{C}}(-)$ is the following composition:

$$\mathcal{C}^{\simeq} \xrightarrow{\Delta} \mathcal{C}^{\simeq} \times \mathcal{C}^{\simeq} \rightarrow \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\mathrm{Hom}} \mathbf{Ani}.$$

As a special case, for an anima $X \in \mathbf{Ani}$ there is a functor

$$\Omega(X, -): X \rightarrow \mathbf{Grp}$$

sending each point x to the *loop group* $\Omega(X, x)$.

Example 3.1.4 (free product and one-point union). Given two groups G and H , their coproduct in \mathbf{Grp} , called the *free product* $G * H$, corresponds to the one-point union $BG \vee BH$ which is the coproduct in $\mathbf{Ani}_{*/}^{\geq 1}$.

3.2. Group action. Let G be a group.

Definition 3.2.1 (group action). A G -action in a category \mathcal{C} is a functor

$$F: BG \rightarrow \mathcal{C}.$$

The object $X := F(*)$ is called the *underlying object* of the G -action, and we say that F is a G -action on X . Without further specification, we assume $\mathcal{C} = \mathbf{Ani}$.

Observation 3.2.2. Since BG is connected, a G -action on an object X is equivalently a pointed map to the connected component $BG \rightarrow B \mathrm{Aut}_{\mathcal{C}}(X)$, which is a group homomorphism $G \rightarrow \mathrm{Aut}_{\mathcal{C}}(X)$. But this is only an object-level description, because $\mathrm{Aut}_{\mathcal{C}}(-)$ is not a functor on \mathcal{C} , only on \mathcal{C}^{\simeq} .

Proposition 3.2.3. The category of G -actions in \mathbf{Ani} is

$$\mathrm{Fun}(BG, \mathbf{Ani}) \simeq \mathbf{Ani}_{/BG} \simeq \mathbf{LMod}(G),$$

where $\mathbf{LMod}(G)$ is the category of left G -modules in \mathbf{Ani} (G regarded as an \mathbb{E}_1 -algebra).

Definition 3.2.4 (restriction). Let $f: H \rightarrow G$ be a group homomorphism. The *restriction* of a G -action $F: BG \rightarrow \mathbf{Ani}$ along f is the H -action given by the composition

$$BH \xrightarrow{Bf} BG \xrightarrow{F} \mathbf{Ani},$$

corresponding to the pullback $(Bf)^*: \mathbf{Ani}_{/BG} \rightarrow \mathbf{Ani}_{/BH}$.

Definition 3.2.5. The *invariants* and *coinvariants* of the G -action are the limit and colimit of the functor F . The invariants is denoted X^G if the action is clear from context. The coinvariants is sometimes also called the *quotient* and denoted X/G .

Example 3.2.6 (action on the point). There is only one G -action on the point $*$, namely the constant functor $BG \rightarrow * \xrightarrow{*} \mathbf{Ani}$. Its quotient is $*/G = BG$.

Proposition 3.2.7. Under the equivalence in Proposition 3.2.3, a G -action on X corresponds to the object

$$(3.1) \quad X/G \rightarrow */G \in \mathbf{Ani}/_{BG}.$$

Moreover, the object X can be recovered by taking the fiber of (3.1):

$$X = \text{fib}(X/G \rightarrow */G);$$

and the invariants is the anima of sections:

$$X^G = \text{Hom}_{\mathbf{Ani}/_{BG}}(*/G, X/G).$$

Observation 3.2.8. Taking the Čech nerve of each row in the pullback diagram

$$\begin{array}{ccc} X & \longrightarrow & X/G \\ \downarrow & & \downarrow \\ * & \longrightarrow & BG \end{array}$$

gives a pullback diagram

$$\begin{array}{ccccccc} \cdots & & X \times_{X/G} X \times_{X/G} X & \rightrightarrows & X \times_{X/G} X & \rightrightarrows & X \longrightarrow X/G \\ & & \downarrow & & \downarrow & & \downarrow & \downarrow \\ \cdots & & G \times G & \rightrightarrows & G & \rightrightarrows & * \longrightarrow BG. \end{array}$$

In particular we have

$$X \times_{X/G} X = X \times G.$$

This identification gives the “action map” $X \times G \rightarrow X$ used to define a group action in classical group theory. But this map alone is no longer sufficient in ∞-group theory.

Example 3.2.9. The *left-right action* of $G \times G$ on G is the Hom-functor

$$\text{Hom}: BG^{\text{op}} \times BG \simeq BG \times BG \rightarrow \mathbf{Ani}.$$

It corresponds to the diagonal $\Delta_{BG}: BG \rightarrow BG \times BG$ in $\mathbf{Ani}/_{BG \times BG}$.

Example 3.2.10. The (right) *regular action* of G on itself is the restriction of the left-right action along $(*, \text{id}): G \rightarrow G \times G$. It corresponds to the object $* \rightarrow */G$ in $\mathbf{Ani}/_{BG}$.

Definition 3.2.11. The *adjoint action* of G on itself is the functor

$$\Omega(BG, -): BG \rightarrow \mathbf{Grp}$$

introduced in Example 3.1.3. If we forget the group structure and only look at the action on the anima G , then the adjoint action is also the restriction of the left-right action along the diagonal $\Delta: G \rightarrow G \times G$.

Proposition 3.2.12. Let H, G be two groups. The mapping space $\text{Hom}_{\mathbf{Ani}}(BH, BG)$ is the quotient of $\text{Hom}_{\mathbf{Grp}}(H, G)$ by the adjoint action of G , i.e. the quotient of the G -action given by

$$BG \xrightarrow{\Omega(BG, -)} \mathbf{Grp} \xrightarrow{\text{Hom}_{\mathbf{Grp}}(H, -)} \mathbf{Ani}.$$

Informally we write

$$\text{Hom}_{\mathbf{Ani}}(BH, BG) = \text{Hom}_{\mathbf{Grp}}(H, G)/G.$$

Proof. For any¹ point $x \in BG$, we have a pullback diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Grp}}(H, \Omega(BG, x)) & \longrightarrow & * \\ \downarrow & & \downarrow x \\ \mathrm{Hom}_{\mathbf{Ani}}(BH, BG) & \longrightarrow & BG. \end{array}$$

This shows that the desired G -action on $\mathrm{Hom}(H, G)$ corresponds to the morphism

$$\mathrm{ev}_*: \mathrm{Hom}_{\mathbf{Ani}}(BH, BG) \rightarrow BG,$$

and in particular, that the quotient of this G -action is $\mathrm{Hom}_{\mathbf{Ani}}(BH, BG)$. \square

3.3. Stabilizer.

Definition 3.3.1. Let X be an anima equipped with a G -action. The *stabilizer* of a point $x \in X$ is the loop group of the quotient:

$$\mathrm{Stab}_G(x) := \Omega(X/G, x).$$

Proposition 3.3.2. The underlying anima of the stabilizer $\mathrm{Stab}_G(x)$ is the fiber of the map $G \rightarrow \mathrm{Aut}(X) \xrightarrow{\mathrm{ev}_x} X$ at x .

Proof.

$$\begin{aligned} \mathrm{Stab}_G(x) &= * \times_{X/G} * \\ &= * \times_X (X \times_{X/G} X) \times_X * \\ &= * \times_X (X \times G) \times_X * \\ &= * \times_{X \times X} (X \times G) \\ &= * \times_X G. \end{aligned}$$

\square

3.4. Group homomorphism. Let $f: H \rightarrow G$ be any group homomorphism, which does not have to be a monomorphism in any sense.

Definition 3.4.1. The *coset space* of f is the quotient of the restriction of the regular action of G along f . Unwinding definitions, it is the fiber of Bf :

$$G/H = \mathrm{fib}(BH \xrightarrow{Bf} BG).$$

Definition 3.4.2. The *kernel* of a group homomorphism $f: H \rightarrow G$ is the homomorphism $\ker(f) \rightarrow H$ in the following pullback square in \mathbf{Grp} (which is also a pullback square in \mathbf{Ani}).

$$\begin{array}{ccc} \ker(f) & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ H & \xrightarrow{f} & G \end{array}$$

Equivalently, the kernel is given by the loop group of the coset space:

$$B\ker(f) = \mathrm{fib}(Bf)_0 = (G/H)_0, \quad \ker(f) = \Omega(G/H, 1).$$

Proposition 3.4.3. There is a fiber sequence of pointed anima

$$\ker(f) \rightarrow H \xrightarrow{f} G \rightarrow G/H \rightarrow BH \xrightarrow{Bf} BG.$$

¹The quantifier “for any” here should be understood as functorial, or as a Π -type in homotopy type theory.

Proof. Just consider the pullback diagram

$$\begin{array}{ccccccc}
 \ker(f) & \longrightarrow & H & \longrightarrow & * & & \\
 \downarrow & & \downarrow f & & \downarrow & & \\
 * & \longrightarrow & G & \longrightarrow & G/H & \longrightarrow & * \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & * & \longrightarrow & BH & \xrightarrow{Bf} & BG.
 \end{array}$$

□

Observation 3.4.4. If f is essentially surjective in \mathbf{Ani} , i.e. is surjective on π_0 , then from the exact sequence of pointed sets

$$\pi_1(BH) \rightarrow \pi_1(BG) \rightarrow \pi_0(G/H) \rightarrow *$$

we have that $\pi_0(G/H)$ is trivial, i.e. G/H is connected; therefore

$$G/H = B\ker(f).$$

Then $H \rightarrow G$ is a principal $\ker(f)$ -bundle classified by the map $G \rightarrow G/H = B\ker(f)$.

4. CENTRAL THINGS

Let $f: H \rightarrow G$ be a group homomorphism.

4.1. Centralizer.

Definition 4.1.1. The *centralizer* of f , denoted $C_G(H)$, is the invariants of the restriction of the adjoint action of G along f :

$$C_G(H) := \lim(BH \xrightarrow{Bf} BG \xrightarrow{\Omega(BG, -)} \mathbf{Grp}).$$

Observation 4.1.2. The centralizer admits by definition a natural homomorphism

$$C_G(H) \rightarrow G.$$

Proposition 4.1.3. The centralizer of f is the automorphism group of f as an object of the functor category or the mapping space:

$$C_G(H) \simeq \mathrm{Aut}_{\mathrm{Hom}(BH, BG)}(f).$$

Proof. In general, for an anima X , a category \mathcal{C} , and two functors $f, g: X \rightarrow \mathcal{C}$, we have²

$$\mathrm{Hom}_{\mathrm{Fun}(X, \mathcal{C})}(f, g) = \lim_{x \in X} \mathrm{Hom}_{\mathcal{C}}(f(x), g(x)).$$

Specializing to our case ($X = BH$, $\mathcal{C} = BG$, $g = f$), we have

$$\begin{aligned}
 \mathrm{Aut}_{\mathrm{Hom}(BH, BG)}(f) &= \lim_{x \in BH} \Omega(BG, f(x)) \\
 &= \lim(BH \xrightarrow{Bf} BG \xrightarrow{\Omega(BG, -)} \mathbf{Grp}) = C_G(H).
 \end{aligned}$$

□

Proposition 4.1.4. The centralizer of f is the stabilizer of f under the adjoint G -action on $\mathrm{Hom}(H, G)$.

Proof. This follows from Propositions 3.2.12 and 4.1.3. □

Example 4.1.5. The centralizer of the trivial subgroup $1 \rightarrow G$ is G itself:

$$C_G(1) = \Omega(BG, *) = G.$$

²This comes from an end construction over X , using the fact that the twisted arrow category of an anima X is equivalent to X itself.

4.2. Center.

Definition 4.2.1. The *center* of G , denoted $Z(G)$, is the invariants of the adjoint action of G , or equivalently the centralizer of id_G . Applying Proposition 4.1.3 to $f = \text{id}_G$, we have

$$Z(G) = \lim(\text{BG} \xrightarrow{\Omega(\text{BG}, -)} \mathbf{Grp}) = \text{Aut}_{\text{Aut}_{\text{Ani}}(\text{BG})}(\text{id}_{\text{BG}}).$$

Observation 4.2.2. By definition, $Z(G)$ is the second loop group of Ani^\sim at BG , and thus acquires an \mathbb{E}_2 -structure.

Proposition 4.2.3. The center of G coincides with the \mathbb{E}_2 -center of G as an \mathbb{E}_1 -algebra:

$$Z(G) \simeq \text{End}_{\text{End}_{\text{Pr}}(\text{LMod}(G))}(\text{id}_{\text{LMod}(G)}).$$

Here Pr is the category of presentable categories, and the morphisms in Pr are cocontinuous functors.

Proof. The Yoneda embedding

$$\text{BG} \rightarrow \text{Psh}(\text{BG}) \simeq \text{LMod}(G)$$

is fully faithful; therefore the functor

$$\text{End}(\text{BG}) \rightarrow \text{Fun}(\text{BG}, \text{LMod}(G)) \simeq \text{End}_{\text{Pr}}(\text{LMod}(G))$$

is also fully faithful. It thus gives an equivalence

$$Z(G) = \text{End}_{\text{End}(\text{BG})}(\text{id}) \xrightarrow{\sim} \text{End}_{\text{End}_{\text{Pr}}(\text{LMod}(G))}(\text{id}_{\text{LMod}(G)}).$$

□

4.3. Commuting homomorphisms.

Definition 4.3.1. Two homomorphisms $f_1: H_1 \rightarrow G$, $f_2: H_2 \rightarrow G$ are said to *commute* if there is a lifting

$$\begin{array}{ccc} H_1 \times H_2 & \dashrightarrow & G \\ \uparrow & \nearrow (f_1, f_2) & \\ H_1 * H_2 & & \end{array}$$

Note that this is an extra *structure* on (f_1, f_2) rather than a *property*.

The centralizer has a universal property regarding commuting homomorphisms.

Proposition 4.3.2. The centralizer $C_G(H)$ of a homomorphism $f: H \rightarrow G$ is the final object of the category of homomorphisms to G that commute with f ; i.e. any homomorphism f' that commutes with f uniquely factors through $C_G(H)$.

Proof. The data of another homomorphism $f': H' \rightarrow G$ commuting with f is equivalent to that of the diagram

$$\begin{array}{ccccc} \text{BH} & \longrightarrow & \text{BH} \times \text{BH}' & \longleftarrow & \text{BH}' \\ & \searrow \text{Bf} & \downarrow \text{---} & \swarrow \text{Bf}' & \\ & & \text{BG} & & \end{array}$$

or equivalently

$$\begin{array}{ccccc} * & \longrightarrow & \text{BH}' & & \\ & \searrow \text{Bf} & \downarrow \text{---} & \swarrow \text{Bf}' & \\ & & \text{Hom}_{\text{Ani}}(\text{BH}, \text{BG}) & \xrightarrow{\text{ev}_*} & \text{BG}. \end{array}$$

However, the connected component of Bf in $\text{Hom}_{\text{Ani}}(\text{BH}, \text{BG})$ is by definition $\text{BC}_G(H)$, so the data are equivalent to just a group homomorphism $H' \rightarrow C_G(H)$ over G . Thus $C_G(H)$ represents homomorphisms to G that commute with f . □

4.4. Central homomorphism.

Definition 4.4.1. A *central homomorphism* between groups is a homomorphism $f: H \rightarrow G$ that commutes with id_G . Intuitively this means an element from H commutes with all of G .

The following is a direct consequence of Proposition 4.3.2, and gives yet another characterization of the center, this time by a universal property.

Proposition 4.4.2. The center $Z(G)$ admits a final central homomorphism to G ; i.e. any central homomorphism to G factors uniquely through $Z(G)$.

5. NORMAL THINGS

Let $f: H \rightarrow G$ be a group homomorphism.

5.1. Normal homomorphism.

Definition 5.1.1. A *normal homomorphism* is one with a structure of the kernel of an essentially surjective group homomorphism.

5.2. Normalizer.

Definition 5.2.1 (informal). The *normalizer* of f , denoted $N_G(H)$, is the stabilizer of f under the G -action on the collection $(\mathbf{Grp}/_G)^\simeq$ of all group homomorphisms to G by conjugation.

Remark 5.2.2. Definition 5.2.1 is deliberately loose, and will be made rigorous soon; the idea here is to mimic classical group theory, where $N_G(H)$ is the stabilizer of f under the G -action on the set of *subgroups* of G .

Note that it would be different to consider the collection of “all group homomorphisms *from* H to G ” instead of “all group homomorphisms to G ”, even though what really matters is the collection of all group homomorphisms from groups *isomorphic to* H to G . The former would result in Proposition 4.1.4. The latter leaves us the freedom of choice of an H' in the component of H in \mathbf{Grp}^\simeq , i.e. $\mathbf{B} \text{Aut}(H)$, which changes everything.

Consider the connected component $X \rightarrow (\mathbf{Grp}/_G)^\simeq$ of f . Explicitly

$$X = \mathbf{B} \text{Aut}_{\mathbf{Grp}/_G}(f).$$

It is (at least intuitively) clear that Definition 5.2.1 can be equivalently stated using the stabilizer of the G -action on X .

Observe that X is functorial in G . Similarly to Proposition 3.2.12, there is a G -action on X given by the adjoint action of G and the quotient is

$$X/G = \mathbf{B} \text{Aut}_{\mathbf{Ani}_{*/,/\mathbf{BG}}}(Bf).$$

Actually for any $x \in \mathbf{BG}$ there is a pullback diagram

$$\begin{array}{ccccc} \mathbf{B} \text{Aut}_{\mathbf{Grp}/_{\Omega(\mathbf{BG},x)}}(f) & \longrightarrow & \mathbf{Grp}/_{\Omega(\mathbf{BG},x)} & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow x \\ \mathbf{B} \text{Aut}_{\mathbf{Ani}_{*/,/\mathbf{BG}}}(Bf) & \longrightarrow & \mathbf{Ani}_{*/,/\mathbf{BG}}^{\geq 1} & \xrightarrow{\circ} & \mathbf{BG} \end{array}$$

where \circ is composition of morphisms $* \rightarrow X$ and $X \rightarrow \mathbf{BG}$.

This justifies the following definition.

Definition 5.2.3. The *normalizer* of f is

$$N_G(H) = \text{Aut}_{\mathbf{Ani}_{*/,/\mathbf{BG}}}(Bf).$$

Observation 5.2.4. There is a natural factorization of f through $N_G(H)$. One can see this either from Definition 5.2.1 (because the adjoint action of H stabilizes $f \in (\mathbf{Grp}/_G)^\simeq$) or from Definition 5.2.3 (consider the map $\mathbf{B}H \rightarrow \mathbf{Ani}_{*/,/\mathbf{BG}}^{\geq 1}$ that maps $y \in \mathbf{B}H$ to $* \xrightarrow{y} \mathbf{B}H \xrightarrow{Bf} \mathbf{BG}$).

Remark 5.2.5. I failed to find a universal property of $N_G(H)$. My first guess was that $H \rightarrow N_G(H) \rightarrow G$ is the universal factorization of f with $H \rightarrow N_G(H)$ normal; but this seems wrong. Consider an \mathbb{E}_2 -group G which satisfies $G = \ker(1 \rightarrow BG)$; then $G \rightarrow 1$ is itself a normal homomorphism, while its normalizer is $N_1(G) = \text{Aut}_{\text{Grp}}(G)$ which can be nontrivial.

5.3. Weyl group.

Proposition 5.3.1. The natural homomorphism $H \rightarrow N_G(H)$ is normal.

Proof. There is an evident pullback diagram

$$\begin{array}{ccc} BH & \longrightarrow & \text{Ani}_{*/, /BG} \\ \downarrow & & \downarrow \\ * & \xrightarrow{\text{B}f} & \text{Ani}_{/BG} \end{array}$$

from which we get

$$H = \ker(\text{Aut}_{\text{Ani}_{*/, /BG}}(\text{B}f) \rightarrow \text{Aut}_{\text{Ani}_{/BG}}(\text{B}f)).$$

□

This justifies the following definition.

Definition 5.3.2. The *Weyl group* of f is the automorphism group of $\text{B}f$ in the category $\text{Ani}_{/BG}$ of all maps to BG . Formally

$$W = \text{Aut}_{\text{Ani}_{/BG}}(\text{B}f).$$

In other words, the Weyl group is the automorphism group of G/H in the category of G -actions.

Proposition 5.3.3. There is a pullback diagram

$$(5.1) \quad \begin{array}{ccccccc} \ker(f) & \longrightarrow & \text{Aut}_{\text{Grp}/G}(f) & \longrightarrow & * & & \\ \downarrow & & \downarrow & & \downarrow & & \\ H & \longrightarrow & N_G(H) = \text{Aut}_{\text{Ani}_{*/, /BG}}(\text{B}f) & \longrightarrow & G & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & W = \text{Aut}_{\text{Ani}_{/BG}}(\text{B}f) & \longrightarrow & G/H & \longrightarrow & BH \\ & & & & \downarrow & & \downarrow \text{B}f \\ & & & & * & \longrightarrow & BG. \end{array}$$

Proof. Essentially every square is discussed in the above.

□