Quillen's 1969 paper explained

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1 Spectra

First we give a quick review of the abstract properties of spectra we will need.

Spectra are stabilized spaces; the ∞-category of spectra is

$$\mathsf{Sp} = \lim(\cdots \xrightarrow{\Omega} \mathsf{Spc}_* \xrightarrow{\Omega} \mathsf{Spc}_*),$$

so a spectrum can be represented by a sequence of pointed spaces $\{X_n\}_{n\geq 0}$ with equivalences $X_n \simeq \Omega X_{n+1}$. A spectrum X has homotopy groups of every integer degree: if X is represented by $\{X_n\}_{n\geq 0}$, then we can define

$$\pi_k(X) := \pi_{n+k}(X_n)\,(n+k \geq 0).$$

The category Sp is a closed symmetric monoidal category with tensor product $\wedge$ inherited from the smash product $\wedge$ on Spc_*, and there are symmetric monoidal functors

$$(\mathsf{Spc}, \times) \xrightarrow{(-)_+} (\mathsf{Spc}_*, \wedge) \xrightarrow{\Sigma^\infty} (\mathsf{Sp}, \wedge).$$

Their composition is denoted Σ^∞_+. Therefore the tensor unit of Sp is the *sphere spectrum* $\mathbb{S} := \Sigma^\infty S^0 = \Sigma^\infty_+(\mathrm{pt})$. The infinite loopspace functor $\Omega^\infty \colon \mathsf{Sp} \to \mathsf{Spc}_*$ is the right adjoint of Σ^∞.

The inner hom $[X, Y]$ in Sp is called the mapping spectrum, and is presented by the sequence $\{\mathrm{Hom}_{\mathsf{Sp}}(\Sigma^{-n}X, Y)\}$. The inner hom and the tensor product satisfy the adjunction relation

$$[X \wedge Y, Z] \simeq [X, [Y, Z]]$$

and in particular

$$\mathrm{Hom}_{\mathsf{Sp}}(X \wedge Y, Z) \simeq \mathrm{Hom}_{\mathsf{Sp}}(X, [Y, Z]).$$

For any spectrum X, we have $[\mathbb{S}, X] \simeq X$. For two spectra X, Y we have

$$\begin{aligned}\Omega^\infty[X, Y] &\simeq \mathrm{Hom}_{\mathsf{Spc}_*}(S^0, \Omega^\infty[X, Y]) \\ &\simeq \mathrm{Hom}_{\mathsf{Sp}}(\mathbb{S}, [X, Y]) \\ &\simeq \mathrm{Hom}_{\mathsf{Sp}}(X, Y).\end{aligned}$$

A *ring spectrum* is a commutative algebra in $(\mathsf{Sp}, \wedge)$.

A spectrum E determines a homology theory E_* and a cohomology theory E^*:

- $E_*(X) := \pi_*(\Sigma^\infty_+ X \wedge E)$,
- $E^*(X) := \pi_{-*}[\Sigma^\infty_+ X, E] \simeq \pi_0 \operatorname{Hom}_{\mathsf{Sp}}(\Sigma^\infty_+ X, \Sigma^* E)$

In particular,

$$E_*(\mathrm{pt}) = \pi_*(E) = E^{-*}(\mathrm{pt}).$$

Sometimes we denote $\pi_*(E)$ by E_* or E^{-*}. For a ring spectrum E, E_* is an ordinary commutative ring.

1.1 Thom spectra

Let $\xi \colon V \to X$ be a vector bundle of rank d. The *Thom space* X^ξ is the (pointed) homotopy type

$$X^\xi := \operatorname{cofiber}(V \setminus X \to V).$$

Example. The Thom space of a trivial vector bundle $X \times \mathbb{R}^d$ is just $\Sigma^d_+ X$.

The *Thom spectrum* of ξ is

$$\operatorname{Th}(\xi) := \Sigma^{-d} \Sigma^\infty X^\xi.$$

Thom spaces and Thom spectra have the following properties.

- (Functoriality) For $f \colon Y \to X$ there are natural maps $Y^{f^*\xi} \to X^\xi$ and $\operatorname{Th}(f^*\xi) \to \operatorname{Th}(\xi)$;
- (Monoidality) $(X \times Y)^{\xi \boxplus \eta} \simeq X^\xi \wedge Y^\eta$, and $\operatorname{Th}(\xi \boxplus \eta) \simeq \operatorname{Th}(\xi) \wedge \operatorname{Th}(\eta)$;
- (Stability) $X^{\xi \oplus \mathbb{R}} \simeq \Sigma X^\xi$, and $\operatorname{Th}(\xi) \oplus \mathbb{R} \simeq \operatorname{Th}(\xi)$.

Since Th is invariant under adding trivial bundles, we may define Thom spectra for virtual bundles, which are locally formal differences of vector bundles. The spectrum MU is the Thom spectrum of the universal virtual bundle on $\mathrm{BU} = \operatorname{colim}_n \mathrm{BU}(n)$. Equivalently, let $\mathrm{MU}(n)$ be the Thom spectrum of the universal bundle on $\mathrm{BU}(n)$, then

$$\mathrm{MU} \simeq \operatorname{colim}_n \mathrm{MU}(n).$$

1.2 Complex oriented ring spectra

For a ring spectrum E we want to assign a *Thom class*

$$U_\xi \in E^{2n}(X^\xi) \simeq \pi_0 \operatorname{Hom}_{\mathsf{Sp}}(\operatorname{Th}(\xi), E)$$

to any complex vector bundle $\xi \colon V \to X$, satisfying the following conditions.

- (Giving orientation on fibers) For each point $x \in X$, $E^{2n}(X^\xi) \to E^{2n}(\mathrm{pt}^\xi) \simeq E^0(\mathrm{pt})$ sends U_ξ to 1;
- (Pullback compatibility) $U_{f^*\xi} = f^* U_\xi$;
- (Multiplicativity) $U_{\xi \oplus \eta} = U_\xi \cdot U_\eta$;

A ring spectrum with the above structure is called a *complex oriented ring spectrum.* To see exactly what we need for such a structure, consider the universal complex line bundle γ on $\mathbb{C}P^\infty$. It should give a class $U_\gamma \in E^2((\mathbb{C}P^\infty)^\gamma)$ that restricts to $1 \in E^2(\mathrm{pt}^\gamma)$.

However, observe that the map $\mathbb{C}P^\infty \to (\mathbb{C}P^\infty)^\gamma$ happens to be an equivalence, because if we remove the zero section from the total space of γ it becomes a contractible space. Thus[1] we can reformulate the desired structure as the following

Definition. A *complex oriented ring spectrum* (E, x_E) is a ring spectrum E with a class $x_E \in E^2(\mathbb{C}P^\infty)$ restricting to $1 \in E^2(\mathbb{C}P^1)$.

Proposition. The ring spectrum MU is complex oriented.

For any complex vector bundle $\xi \colon V \to X$, let $\chi \colon X \to \mathrm{BU}$ be its classifying map, then the functoriality of Th gives a map

$$\mathrm{Th}(\xi) \to \mathrm{MU}.$$

1.3 Chern classes

Let (E, x_E) be a complex oriented ring spectrum. Let $L \to X$ be a complex line bundle with classifying map $\chi \colon X \to \mathbb{C}P^\infty$. The *first Chern class* of L is

$$c_1^E(L) := \chi^* x_E \in E^2(X).$$

2 Complex cobordism and MU

A *stable almost complex structure* on a manifold M is a complex structure on $TM \oplus \mathbb{R}^N$ for N sufficiently large.

Theorem (Pontryagin–Thom). MU_* is the ring of bordism classes of manifolds with stable almost complex structure. More generally, $\mathrm{MU}_n(X)$ is the bordism group of n-manifolds with a stable almost complex structure and a map to X.

[1] There remains a problem as to determine what the relevant map $\mathbb{C}P^1 \to \mathbb{C}P^\infty$ is. I don't know why, but we assume that this map is just the usual inclusion.

3 Formal group laws

3.1 The formal affine line

A *formal group law* over a commutative ring R is a (commutative) group structure on the formal affine line $\widehat{\mathbb{A}}^1_R$ with the origin being the unit. Concretely, it is a map $F\colon \widehat{\mathbb{A}}^1_R \times \widehat{\mathbb{A}}^1_R \to \widehat{\mathbb{A}}^1_R$, i.e. a two-variable formal power series $F(X,Y)$, satisfying

- $F(X,0) = X = F(0,X)$,
- $F(F(X,Y),Z) = F(X,F(Y,Z))$,
- $F(X,Y) = F(Y,X)$.

The *invariant differential form* of a formal group law F is

$$\omega(X) := \frac{dX}{\partial_2 F(X,0)}.$$

3.2 Logarithm

A *logarithm* of a formal group law F is a series $\ell(X) = X + a_2X^2 + \cdots$ over $R \times \mathbb{Q}$ such that

$$\ell(F(X,Y)) = \ell(X) + \ell(Y).$$

It defines an isomorphism from F (after tensoring with $\mathbb{Q}$) to the additive formal group law.

An important result in the paper is the logarithm of the formal group law associated with complex cobordism theory.

Taking the derivative at $Y = 0$, we get

$$\ell'(X)\partial_2 F(X,0) = 1.$$

Thus the logarithm can be determined by

$$\ell'(X)dX = \omega(X), \quad \ell(0) = 0.$$

3.3 Curves

In this context, a *curve* refers to a self-map of $\widehat{\mathbb{A}}^1_R$ preserving the origin, or equivalently a formal power series

$$f(X) = a_1X + a_2X^2 + \cdots$$

with no constant term.

The group structure F on $\widehat{\mathbb{A}}^1_R$ induces operations on curves:

$$(f +_F g)(X) := F(f(X), g(X)).$$

4 The formal group law of a complex oriented ring spectrum

A complex oriented ring spectrum (E, x_E) defines a formal group law F^E over $E^*(\mathrm{pt})$ satisfies the relation

$$F^E(c_1^E(L_1), c_1^E(L_2)) = c_1^E(L_1 \otimes L_2)$$

where L_1, L_2 are arbitrary complex line bundles on a space X.

To define F^E, consider the universal *pair* of complex line bundles, which is the two pullbacks of the universal complex line bundle γ on $\mathbb{C}P^\infty$ to $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$. Their tensor product is the line bundle $\gamma \boxtimes \gamma$ on $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$. Let $\chi\colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ be the classifying map of $\gamma \boxtimes \gamma$. Since $E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = E^*(\mathrm{pt})[[X, Y]]$ (X, Y being on degree 2), the pullback $\chi^* x_E \in E^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ is a power series $\sum a_{k\ell} X^k Y^\ell$ with $a_{kl} \in E^{2-2k-2\ell}(\mathrm{pt})$.

The input to Qullen's paper is the following special case of a theorem.

Theorem. Let L be the canonical line bundle on $\mathbb{C}P^n$ and u be a polynomial over MU_*. Then the Gysin homomorphism $\pi_*\colon \mathrm{MU}^q(\mathbb{C}P^n) \to \mathrm{MU}^{q-2n}(\mathrm{pt})$ is given by

$$\pi_*(u(c_1^{\mathrm{MU}}(L))) = \operatorname{res} \frac{u(X)\omega(X)}{X^{n+1}}$$

Taking $u = 1$, we get

Corollary. The coefficient of $X^n dX$ in $\omega(X)$ is $[\mathbb{C}P^n]$.

Corollary. The logarithm of F^{MU} is

$$\ell(X) = \sum_{n \geq 0} \frac{[\mathbb{C}P^n]}{n+1} X^{n+1}.$$

5 The universal nature of cobordism group laws

Theorem. For any formal group F over a commutative ring R, there is a unique homomorphism $\mathrm{MU}^*(\mathrm{pt}) \to R$ carrying F^{MU} to F.

Quillen's proof of the above result uses two previously known facts:

- Lazard (1955) had determined the ring L over which the universal formal group law is defined,
$$L \simeq \mathbb{Z}[x_1, x_2, \cdots]/;$$
- Milnor and Novikov (1960) had determined the structure of the ring MU_*,
$$\mathrm{MU}_* \simeq \mathbb{Z}[x_1, x_2, \cdots]\,(\deg(x_i) = 2i).$$

Given Lazard's result, the theorem can be stated as follows: the ring homomorphism $h\colon L \to \mathrm{MU}_*$ giving F^{MU} is an isomorphism.

Proof that $h \otimes \mathbb{Q}$ is an isomorphism.

- $L \otimes \mathbb{Q} \simeq \mathbb{Q}[p_1, p_2, \cdots]$.
- $h(p_i) = [\mathbb{C}P^i]$.
- $\mathrm{MU}_* \otimes \mathbb{Q} \simeq \mathbb{Q}[[\mathbb{C}P^1], [\mathbb{C}P^2], \cdots]$.

Suppose the logarithm of the universal formal group law over L is

$$\sum_{n \geq 0} \frac{p_n}{n+1} X^{n+1}.$$

Then the law over $L \otimes \mathbb{Q}$ is universal for laws over $\mathbb{Q}$-algebras. Observe that any law over a $\mathbb{Q}$-algebra is uniquely determined by its logarithm which can be any series with leading term X; so h sends p_i to $[\mathbb{C}P^i]$ and $h \otimes \mathbb{Q}$ is an isomorphism.

Proof that h is an injection.

- $L \simeq \mathbb{Z}[x_1, x_2, \cdots]$.
- In particular, L is torsion-free.

Proof that h is a surjection.

- $p_n \in L$ (because p_n is the coefficient of X^n in), so $[\mathbb{C}P^n] \in h(L)$.
- For
$$M_n \subset \mathbb{C}P^{n_1} \times \cdots \times \mathbb{C}P^{n_r}$$
a nonsingular hypersurface of degree $k_1, \cdots, k_r$, $[M_n]$ is contained in the image of h.

Denote by π the map from M_n to a point, and by L_j the pullback of the tautological line bundle on the j-th factor. Then

$$\begin{aligned}[M_n] &= \pi_* c_1^{\mathrm{MU}}(L_1^{k_1} \otimes \cdots \otimes L_r^{k_r}) \text{ (using the input lemma on Gysin homomorphism)} \\ &= \pi_*\big(k_1 c_1^{\mathrm{MU}}(L_1) + \cdots + k_r c_1^{\mathrm{MU}}(L_r)\big) \text{ (in the sense of } F_{\mathrm{MU}}) \\ &= \pi_* \sum_{i_1, \cdots, i_r} \pi^*(a_{i_1 \cdots i_r}) c_1^{\mathrm{MU}}(L_1)^{i_1} \cdots c_r^{\mathrm{MU}}(L_r)^{i_r}, a_{i_1 \cdots i_r} \in h(L)\end{aligned}$$

Since

$$\pi_* c_1^{\mathrm{MU}}(L_1)^{i_1} \cdots c_r^{\mathrm{MU}}(L_r)^{i_r} = \prod_{j=1}^{r} [\mathbb{C}P^{n_j - i_j}]$$

is also in $h(L)$, it follows that $[M_n] \in h(L)$.

6 p-typicality

Given a formal group law F, a curve $f(X)$ and a positive integer n, let

$$(F_n f)(X) = \sum_{i=1}^{n}{}^{F} f(\zeta_i X^{1/n}).$$

Here $\sum^F$ means addition defined by the formal group law F, and ζ_i are the n-th roots of 1. A priori this is a power series over $R[\zeta]$, but by symmetry it is actually over R.

Example. When F is the additive law and $f(X) = a_1X + a_2X^2 + \cdots$, we have $(F_n f)(X) = na_nX + 2na_{2n}X^2 + \cdots$.

Definition. Given a formal group law F, a curve f is called p-typical if $F_q f = 0$ for every prime $q \neq p$. A formal group itself is called p-typical if the curve $f(X) = X$ is p-typical.

Example. When F is the additive law, a curve f is p-typical iff $f(X)$ has the form $a_pX^p + a_{p^2}X^{p^2} + \cdots$.

If R is a $\mathbb{Z}_{(p)}$-algebra and is torsion-free, then a curve f is p-typical iff the series $\ell(f(X))$ over $R \otimes \mathbb{Q}$ has only terms of degree a power of p, where ℓ is the logarithm of F.

7 Decomposition of $\mathrm{MU}_{(p)}$

In 1966 Brown and Peterson showed that after localization at a prime p, MU splits into a wedge of smaller spectra now known as BP and denoted by Quillen as ΩT. This splitting is suggested by a corresponding decomposition of $H^*\mathrm{MU}$ as a module over the mod p Steenrod algebra.

Quillen gave a much cleaner form of the splitting using some algebra developed by Pierre Cartier. He thereby showed that BP is a ring spectrum.

Cartier showed that when R is a $\mathbb{Z}_{(p)}$-algebra, there is a canonical coordinate change that converts any formal group law into a p-typical one. Quillen used this to define an idempotent map $\widehat{\xi}$ on $\mathrm{MU}_{(p)} = \mathrm{MU} \otimes \mathbb{Z}_{(p)}$ whose telescope is BP.

This process changes the logarithm from

$$\sum_{n\geq 0} \frac{[\mathbb{C}P^n]}{n+1} X^{n+1} \quad \text{to} \quad \sum_{k\geq 0} \frac{[\mathbb{C}P^{p^k-1}]}{p^k} X^{p^k}.$$

Fix a prime p and let R be a $\mathbb{Z}_{(p)}$-algebra, meaning that we can divide by any integer n prime to p.

Theorem (Cartier). A formal group law over R is canonically strictly isomorphic to a p-typical one.

To prove this, it suffices to construct a strict isomorphism from the law over $L \otimes \mathbb{Z}_{(p)}$ to a typical law.

Construction. Let c_F be the curve given by

$$c_F^{-1} = \sum_{(n,p)=1} \frac{\mu(n)}{n} V_n F_n \gamma_0,$$

where γ_0 is the curve $\gamma_0(X) = X$, V_n is the operation taking a curve f to $f(X^n)$, the sum as well as division by n prime to p is taken in the filtered group of curves and μ is the Möbius function.

Proposition. The group law

$$(c_{F*}F)(X,Y) = c_F(F(c_F^{-1}X, c_F^{-1}Y))$$

is typical.

Proposition-definition. Let $\xi = c_{F^{\mathrm{MU}}}$; it is a power series over $\mathrm{MU}^*_{(p)}$ with leading term X. There is a unique multiplicative natural transformation (i.e. homomorphism of ring spectra) $\widehat{\xi}\colon \mathrm{MU}^*_{(p)}(-) \to \mathrm{MU}^*_{(p)}(-)$ such that

$$\widehat{\xi} c_1^{\mathrm{MU}}(L) = \xi(c_1^{\mathrm{MU}}(L))$$

for all line bundles L.

Proposition. The operation $\widehat{\xi}$ is idempotent, and its values on $\mathrm{MU}^*_{(p)}(\mathrm{pt})$ are

$$\widehat{\xi}([\mathbb{C}P^n]) = \begin{cases} [\mathbb{C}P^n] & \text{if } n = p^a - 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathrm{BP}^*(X)$ be the image of $\widehat{\xi}$. Then the following are pushout diagrams of commutative rings.

$$\begin{array}{ccc} \mathrm{MU}^*_{(p)}(\mathrm{pt}) & \longrightarrow & \mathrm{BP}^*(\mathrm{pt}) \\ \downarrow & & \downarrow \\ \mathrm{MU}^*_{(p)}(X) & \longrightarrow & \mathrm{BP}^*(X) \end{array} \qquad \begin{array}{ccc} \mathrm{MU}^*_{(p)}(\mathrm{pt}) & \longleftarrow & \mathrm{BP}^*(\mathrm{pt}) \\ \downarrow & & \downarrow \\ \mathrm{MU}^*_{(p)}(X) & \longleftarrow & \mathrm{BP}^*(X) \end{array}$$

8 Operations on BP*

In two pages, Quillen gave a precise description of the graded algebra of maps from BP to itself.

The surjection $\widehat{\xi}\colon \mathrm{MU}^*_{(p)} \to \mathrm{BP}^*$ carries the Thom class in $\mathrm{MU}^*_{(p)}(\mathrm{MU})$ to one for BP^*. As a consequence BP has the usual machinery of characteristic classes with $c_i^{\mathrm{BP}}(-) = \widehat{\xi} c_i^{\mathrm{MU}}(-)$, and $f^{\mathrm{BP}} = \widehat{\xi} F^{\mathrm{MU}}$.

Let $t_1, t_2, \cdots$ be formal variables and $t_0 = 1$. Consider the series

$$\phi_t(X) = \sum_{n\geq 0}{}^{F^{\mathrm{BP}}} t_n X^{p^n}.$$

There is a unique morphism of ring spectra $\widehat{\phi_t^{-1}}\colon \mathrm{MU} \to \mathrm{BP}[t_1, t_2, \cdots]$ such that

$$\widehat{\phi_t^{-1}}(c_1^{\mathrm{MU}}(L)) = \phi_t^{-1}(c_1^{\mathrm{BP}}(L))$$

for all line bundles L.

Writing

$$r_t(x) = \sum_\alpha r_\alpha(x) t^\alpha \qquad \text{if } x \in \Omega T^*(X)$$

where the sum is taken over all sequences $\alpha = (\alpha_1, \alpha_2, \cdots)$ of natural numbers all but a finite number of which are zero, we obtain stable operations

$$r_\alpha\colon \Omega T^*(X) \to \Omega T^*(X).$$

THEOREM 5. (i) *r_α is a stable operation of degree $2\sum_i \alpha_i(p^i - 1)$. Every stable operation may be uniquely written as an infinite sum*

$$\sum_\alpha u_\alpha r_\alpha \qquad u_\alpha \in \Omega T^*(pt)$$

and every such sum defines a stable operation.

(ii) *If $x, y \in \Omega T^*(X)$, then*

$$r_\alpha(xy) = \sum_{\beta+\gamma=\alpha} r_\beta(x) r_\gamma(y).$$

(iii) *The action of r_α on $\Omega T^*(pt)$ is given by*

$$r_t(P_{p^n-1}) = \sum_{h=0}^{n} p^{n-h} P_{p^h-1} t_{n-h}^{p^h}.$$

(iv) *If $t' = (t_1', t_2', \cdots)$ is another sequence of indeterminates, then the compositions $r_\alpha \circ r_\beta$ are found by comparing the coefficients of $t^\alpha t'^\beta$ in*

$$r_t \circ r_{t'} = \sum_\gamma \Phi(t, t')^\gamma r_\gamma$$

where $\Phi = (\Phi_1(t_1; t_1'), \Phi_2 = (t_1, t_2; t_1', t_2'), \cdots)$ is the sequence of polynomials with coefficients in $\Omega T^*(pt)$ in the variables t_i and t_i' obtained by solving the equations

$$\sum_{h=0}^{N} p^{N-h} P_{p^h-1} \Phi_{N-h}^{p^h} = \sum_{k+m+n=N} p^{m+n} P_{p^k-1} t_m^{p^k} t_n'^{p^{k+m}}.$$