Principal ∞-bundles and ∞-topos theory

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Outline

The main reference is *Principal ∞-bundles — General theory* by Thomas Nikolaus, Urs Schreiber, Danny Stevenson.

Outline

Conventions

We use an intrinsic and model-independent language for ∞-categories, in which

- a 1-category is automatically regarded as an ∞-category.
- all functors, limits and colimits are ∞-category-theoretical.

Some notations:

- $\mathbf{H}$ denotes a fixed (Grothendieck) ∞-topos;
- $\mathbf{H}^{\rightarrow}$ denotes the arrow category $\mathrm{Fun}(\{* \to *\}, \mathbf{H})$;
- $\mathbf{H}^{*/}$ denotes the category of pointed objects of $\mathbf{H}$;
- $\mathbf{H}^{*/}_{\geq 1}$ the category of pointed connected objects of $\mathbf{H}$.

Section 2

Introduction to ∞-toposes

Introduction to ∞-toposes

- A topos is a category in which we can do *set theory*; we can perform in a topos everything we do on sets.
- Subsets from logical formulas, quotients by equivalence relations, sets of functions, ...
- The archetypical topos is Set.

- An ∞-topos is an ∞-category in which we can do *homotopy theory*; we can perform in an ∞-topos everything we do on spaces.
- Homotopy groups, cohomology, connectivity, truncatedness, Postnikov towers, delooping, stabilization, ...
- The archetypical ∞-topos is Grpd_∞.
- The full subcategory of $(n-1)$-truncated objects in an ∞-topos form an *n-topos*.

Grothendieck toposes

Definition (Grothendieck topos)

A *Grothendieck topos* is a left exact localization[1] of a presheaf category.

The definition is verbatim the same in ∞-category theory and gives Grothendieck ∞-toposes. Note that a presheaf category in this context is $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Grpd}_\infty)$.

[1] A left exact localization is a fully faithful functor with a left adjoint that preserves finite limits.

Giraud axioms

The Giraud [ʒiʁo] axioms are a characterization of toposes in terms of category-theoretical properties.

Definition (Giraud axioms)

A presentable category $\mathcal{C}$ is said to satisfy the *Giraud axioms* if

1. colimits in $\mathcal{C}$ are universal (i.e. stable under pullback);
2. coproducts in $\mathcal{C}$ are disjoint;
3. quotients in $\mathcal{C}$ are effective epimorphisms (i.e. coequalizers).

These conditions all have analogs in ∞-category theory, and form the ∞-*Giraud axioms*. To state the third condition in ∞-categorical terms, we need *groupoid objects*.

Groupoids and groups

Definition (groupoid object)

A *groupoid object* in an ∞-category $\mathcal{C}$ is a simplicial object $\mathcal{G}\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ such that the natural map

$$\mathcal{G}_n \to \mathcal{G}_1 \times_{\mathcal{G}_0} \times \cdots \times_{\mathcal{G}_0} \mathcal{G}_1$$

is an equivalence for every $n \geq 1$. The object $\mathcal{G}_0$ is the "set of vertices" of the groupoid. We also call $\mathcal{G}$ a groupoid object *over* $\mathcal{G}_0$. Denote by $\mathrm{Grpd}(\mathcal{C})$ the full subcategory of $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ spanned by groupoid objects.

Groupoids and groups

Definition (group object)

A *group object* [2] in an ∞-category $\mathcal{C}$ is a groupoid object $\mathcal{G}$ with an equivalence

$$\mathcal{G}_0 \simeq 1.$$

We call $\mathcal{G}_1$ the *underlying object* of $\mathcal{G}$, and denote it by G. By abuse of notation we also speak of a *group object* G.
Denote by $\mathrm{Grp}(\mathcal{C})$ the full subcategory of $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ spanned by group objects.

[2] This involves a special case of the *delooping hypothesis*. Group objects can also be defined as A_∞-algebras, and by delooping, they are equivalent to this definition (at least in ∞-toposes).

Groupoids and groups

The concept of group(-oid) objects in an ∞-category generalizes that of ordinary group(-oid) objects in a 1-category.

- A group object in Set is an ordinary group.
- A group object in Mfd is a Lie group.
- A group object in Grpd_∞ is an ∞-*group*, or equivalently an A_∞-*algebra*.
- A groupoid object in $\mathrm{Sh}(\mathrm{Mfd}, \mathrm{Grpd}_\infty)$ is called a *smooth* ∞-*groupoid*.

Groupoids and groups

For $\mathcal{G}$ a groupoid object, the object $\operatorname{colim} \mathcal{G}$ may be thought of as a quotient of $\mathcal{G}_0$ obtained by gluing along the morphisms of $\mathcal{G}$.

Definition (quotient projection of a groupoid object)

For a groupoid object $\mathcal{G}$ in an ∞-topos $\mathbf{H}$, The natural map to the colimit

$$\mathcal{G}_0 \to \operatorname{colim} \mathcal{G}$$

is called the *quotient projection* of the groupoid object.

Groupoids and groups

- By definition of colimits, there is an adjunction

$$\mathbf{H} \underset{\text{const}}{\overset{\text{colim}}{\underset{\bot}{\leftrightarrows}}} \mathrm{Grpd}(\mathbf{H})$$

  that exhibits $\mathbf{H}$ as a reflective subcategory of $\mathrm{Grpd}(\mathbf{H})$.
- The functor colim in this context is also called the *realization*.

Groupoids and groups

Definition (Čech nerve, effective epimorphism)

Let $\mathcal{C}$ be an ∞-category with pullback. To any morphism $P \to X$ in $\mathcal{C}$ is associated a groupoid object

$$\check{C}(P \to X) = \cdots \substack{\to\\\to\\\to\\\to} P \times_X P \times_X P \substack{\to\\\to\\\to} P \times_X P \rightrightarrows P$$

called the *Čech nerve*.

A morphism $P \to X$ is called an *effective epimorphism* if it is the quotient projection of its own Čech nerve:

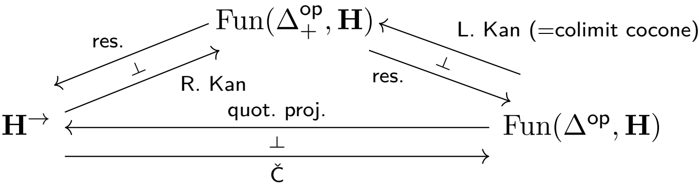
$$\operatorname{colim} \check{C}(P \to X) \xrightarrow{\simeq} X.$$

Čech nerve and quotient projection

The Čech nerve as a right Kan extension:

$$\Delta_+ = \left\{ \cdots [1] \leftleftarrows [0] \leftarrow [-1] \right\}$$

$$\left\{ [0] \leftarrow [-1] \right\} \hookrightarrow \Delta_+ \hookleftarrow \Delta = \left\{ \cdots [1] \leftleftarrows [0] \right\}$$



Čech nerve and quotient projection

Here are some side remarks.

- In general, for any morphism $f\colon U \to X$, $\operatorname{colim} \check{C}(f) \to X$ is called the *image* of f. The sequence $U \to \operatorname{im}(f) \to X$ is called the *(epi, mono)-factorization* of f, a special case of the *(n-connected, n-truncated)-factorization* when $n = -1$.
- $\operatorname{im}(f)$ is the (-1)-truncationof f in $\mathcal{C}_{/X}$. In other words, it is the smallest subobject of X through which f can factor.
- Of course, f is an effective epimorphism if and only if $\operatorname{im} f = \operatorname{id}_X$.

Giraud axioms for ∞-toposes

Definition

A presentable ∞-category $\mathcal{C}$ is said to satisfy the *Giraud axioms* if

1. colimits in $\mathcal{C}$ are universal;
2. coproducts in $\mathcal{C}$ are disjoint;
3. every groupoid object in $\mathcal{C}$ is equivalent to the Čech nerve of its quotient projection.

Theorem (HTT 6.1.0.6)

An ∞-category is an ∞-topos if and only if it satisfies the Giraud axioms.

Giraud axioms for ∞-toposes

- The third Giraud axiom requires the Čech nerve–quotient projection adjunction to restrict to an equivalence of full subcategories:

$$\begin{array}{ccc} \mathbf{H}^{\to} & \underset{\check{\mathrm{C}}}{\overset{\text{quot. proj.}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} & \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{H}) \\ \uparrow & & \uparrow \\ \mathbf{H}^{\to}_{\mathrm{eff}} & \underset{\longrightarrow}{\overset{\longleftarrow}{\simeq}} & \mathrm{Grpd}(\mathbf{H}), \end{array}$$

the two subcategories being the *essential images* of the two adjoint functors.

Looping and delooping

- For any pointed object $X \in \mathbf{H}^{*/}$, the loopspace $\Omega X = * \times_X *$ admits a *group structure* given by the *Čech nerve* of $* \to X$,

$$\Omega X = \check{\mathrm{C}}(* \to X), \;\; (\Omega X)_n = * \times_X \cdots \times_X * \, (n+1 \text{ points}).$$

- The loopspace functor $\Omega \colon \mathbf{H}^{*/} \to \mathrm{Grp}(\mathbf{H})$ fits into the diagram

$$\begin{array}{ccc} \mathbf{H}^{\to} & \xrightarrow{\check{\mathrm{C}}} & \mathrm{Grpd}(\mathbf{H}) \\ \uparrow & & \uparrow \\ \mathbf{H}^{*/} & \xrightarrow[\Omega]{} & \mathrm{Grp}(\mathbf{H}). \end{array}$$

Looping and delooping

Lemma (proved in HTT 7.2.2.11)

Let $X \in \mathbf{H}^{*/}$ be a pointed object. The map $* \to X$ is an effective epimorphism if and only if X is connected. In particular we have a full subcategory

$$\mathbf{H}^{*/}_{\geq 1} \hookrightarrow \mathbf{H}^{\to}_{\mathsf{eff}}.$$

Lemma (HTT 7.2.1.14)

A morphism $f\colon X \to Y$ in an ∞-topos $\mathbf{H}$ is an effective epimorphism if and only if its truncation $\tau_{\leq 0} f\colon \tau_{\leq 0} X \to \tau_{\leq 0} Y$ is an effective epimorphism in the underlying 1-topos $\mathrm{h}\tau_{\leq 0}\mathbf{H}$.

- In particular, a map $f\colon X \to Y$ in Grpd_∞ is an effective epimorphism if and only if $\pi_0 f\colon \pi_0 X \to \pi_0 Y$ is a surjection.

Looping and delooping

The functor Ω restricted to $\mathbf{H}^{*/}_{\geq 1} \hookrightarrow \mathbf{H}^{\to}_{\text{eff}}$ gives an equivalence. Its inverse $\mathbf{B}$ is called *delooping*.

$$\begin{array}{ccc} \mathbf{H}^{\to}_{\text{eff}} & \underset{\check{\text{C}}}{\overset{\text{quot. proj.}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq}}}} & \mathrm{Grpd}(\mathbf{H}) \\ \uparrow & & \uparrow \\ \mathbf{H}^{*/}_{\geq 1} & \underset{\Omega}{\overset{\mathbf{B}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq}}}} & \mathrm{Grp}(\mathbf{H}) \end{array}$$

The equivalence $G \simeq \Omega \mathbf{B} G$ gives a fiber sequence

$$G \to * \to \mathbf{B}G.$$

Section 3

Classifying Spaces

Classifying Spaces

Fix an ∞-topos $\mathbf{H}$. Let X be an object of $\mathbf{H}$ and G be a group object of $\mathbf{H}$. The aim of this section is to state and prove the equivalence (of ∞-groupoids)

$$G\mathrm{Bund}(X) \simeq \mathbf{H}(X, \mathbf{B}G).$$

G-actions

Definition (G-action)

A *G-action* on an object P is a groupoid object $P/\!/G$ of the form

$$P/\!/G = \quad \cdots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} P \times G \times G \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} P \times G \underset{\mathrm{pr}_1}{\overset{\rho}{\rightrightarrows}} P$$

such that the projection maps

$$\begin{array}{ccccccc} \cdots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} & P \times G \times G & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & P \times G & \rightrightarrows & P \\ & \downarrow & & \downarrow & & \downarrow \\ \cdots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} & * \times G \times G & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & * \times G & \rightrightarrows & * \end{array}$$

constitute a morphism $P/\!/G \to */\!/G$ of groupoid objects.
The corresponding *quotient object* of the G-action is the colimit of the groupoid object.

G-actions

Definition

The G-actions form a full subcategory

$$G\mathrm{Act}(\mathbf{H}) \hookrightarrow \mathrm{Grpd}(\mathbf{H})_{/(*/\!/G)}.$$

- The groupoid object $*/\!/G$ is G "itself" (regarded as a groupoid object over $*$). Its quotient object of $*/\!/G$ is by definition $\mathbf{B}G$.

Principal G-bundles

Definition (principal G-bundle)

A *principal G-bundle* over X is a morphism $P \to X$ with a G-action on P whose quotient object is X.

- The traditional requirement of *freeness* is not a characterization of principality, but a condition that ensures that the base is a 0-*truncated object*. In the context of ∞-topos, every action is a principal bundle over its quotient.
- Local triviality is also automatic: a principal G-bundle $P \to X$ is always a cover (effective epimorphism), and it trivializes itself. We will see later.

Principal G-bundles

- The Giraud axiom dictates that, for any principal G-bundle $P \to X$ there is an equivalence of groupoid objects

$$P /\!/ G \simeq \check{\mathrm{C}}(P \to X).$$

  This corresponds to the traditional requirement that the *shear map* $P \times G \to P \times_X P$ is an equivalence.

Principal G-bundles

Definition

The category of principal G-bundles over X is

$$G\mathrm{Bund}(X) := G\mathrm{Act}(\mathbf{H}) \times_{\mathbf{H}} \{X\}.$$

Classifying maps

Proposition

For $f\colon X \to \mathbf{B}G$ any morphism, its (homotopy) fiber $P \to X$ is canonically a G-principal bundle.

- Pull back $(*/\!/G) \to \mathbf{B}G$ along f to get $(P/\!/G) \to X$.

$$\begin{array}{ccccccccc}
\cdots & \substack{\to\\\to\\\to\\\to} & P \times G \times G & \substack{\to\\\to\\\to} & P \times G & \substack{\to\\\to} & P & \longrightarrow & X \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\
\cdots & \substack{\to\\\to\\\to\\\to} & * \times G \times G & \substack{\to\\\to\\\to} & * \times G & \substack{\to\\\to} & * & \longrightarrow & \mathbf{B}G
\end{array}$$

$$\Delta^{\mathrm{op}} \xrightarrow{*/\!/G} \mathbf{H} \xrightarrow{p_*} \mathbf{H}_{/\mathbf{B}G} \xrightarrow{f^*} \mathbf{H}_{/X}, \qquad \Delta^{\mathrm{op}} \xrightarrow{P/\!/G} \mathbf{H}_{/X}$$

- Using the fact (first Giraud axiom) that f^* preserves colimits, we have $\mathrm{colim}\, P/\!/G \simeq X$, so $P \to X$ is a G-principal bundle.

Classifying maps

- This defines a map

$$\mathbf{H}(X, \mathbf{B}G) \to G\mathrm{Bund}(X).$$

Trivial G-bundles

Definition (trivial G-bundle)

The trivial G-bundle on X is the one obtained from the morphism $X \to * \to \mathbf{B}G$.

$$\begin{array}{ccccc} X \times G & \longrightarrow & G & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & * & \longrightarrow & \mathbf{B}G \end{array}$$

Local triviality of principal G-bundles

Proposition (local triviality)

For any principal G-bundle $P \to X$, there exists an effective epimorphism $U \to X$ such that the pullback bundle of $P \to X$ over U is trivial.

- The proof just takes $U = P$. What's important is the existence of the cover $U \to X$, not its actual structure.

Classifying maps

Every principal G-bundle on X comes from a pullback along $X \to \mathbf{B}G$.

Proposition

For every principal G-bundle $P \to X$ the rightmost square in

$$\begin{array}{ccccccccc} \cdots & \substack{\to\\\to\\\to\\\to} & P \times G \times G & \substack{\to\\\to\\\to} & P \times G & \rightrightarrows & P & \longrightarrow & X \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow f \\ \cdots & \substack{\to\\\to\\\to\\\to} & * \times G \times G & \substack{\to\\\to\\\to} & * \times G & \rightrightarrows & * & \longrightarrow & \mathbf{B}G \end{array}$$

is a pullback.

- The proof uses a “local-to-global” argument.

Classifying maps

Lemma (effective epis reflect equivalences)

Consider the following pullback diagram in an ∞-topos.

$$\begin{array}{ccc} A & \overset{\varphi}{\twoheadrightarrow} & B \\ {\scriptstyle\simeq}\downarrow & & \downarrow \\ A' & \underset{\varphi'}{\twoheadrightarrow} & B' \end{array}$$

Suppose φ' (and moreover, φ) is an effective epimorphism, and the left map is an equivalence, then the right map is also an equivalence.

- Intuition: φ is a cover, and the condition says B is *locally* equivalent to B'.

Classifying maps

- We take the Čech nerves Č(φ) and Č(φ'):

$$\begin{array}{ccccccccc}
\cdots \; \overrightarrow{\rightrightarrows\rightrightarrows} & A \times_B A \times_B A & \overrightarrow{\rightrightarrows} & A \times_B A & \rightrightarrows & A & \xrightarrow{\varphi} & B \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots \; \overrightarrow{\rightrightarrows\rightrightarrows} & A' \times_{B'} A' \times_{B'} A' & \overrightarrow{\rightrightarrows} & A' \times_{B'} A' & \rightrightarrows & A' & \xrightarrow[\varphi']{} & B'
\end{array}$$

- All vertical arrows to the left of $A \to A'$ are pullbacks of $A \to A'$, and are thus equivalences. The conclusion follows from the functoriality of colimits.

Classifying maps

Lemma (effective epis reflect pullbacks)

Consider the following diagram in an ∞-topos.

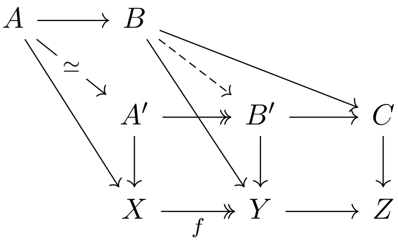
$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ X & \underset{f}{\twoheadrightarrow} & Y & \longrightarrow & Z \end{array}$$

Suppose f is an effective epimorphism, and the left square and the outer rectangle are pullbacks. Then the right square is also a pullback.

- Intuition: f is a cover, and the condition says B is a pullback *locally* on Y.

Classifying maps

- Take $B' = Y \times_Z C$ and $A' = X \times_Y B'$:
  (all parallelograms are pullbacks)



- $A \to A'$ is an equivalence.
- $A' \to B'$ is an effective epimorphism.
- The previous lemma then implies that $B \to B'$ is an equivalence.

Classifying maps

- Applying the lemma to the diagram

$$\begin{array}{ccc} U \times G & \longrightarrow & U \\ \downarrow & & \downarrow\!\!\downarrow \\ P & \longrightarrow & X \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbf{B}G \end{array}$$

gives the conclusion.

Classifying maps

We have seen an equivalence

$$G\mathrm{Bund} \simeq \mathrm{Cart}(\mathbf{H})^{\to}_{/(*\to\mathbf{B}G)} \times_{\mathbf{H}} \{X\}.$$

The latter is equivalent to $\mathbf{H}_{/\mathbf{B}G} \times_{\mathbf{H}} \{X\}$, which is in turn equivalent to $\mathbf{H}(X, \mathbf{B}G)$.

Section 4

Applications

Group extensions

- Fix a group G and an abelian group A. A *group extension* of G by A is a short exact sequence of groups

$$1 \to A \hookrightarrow E \twoheadrightarrow G \to 1.$$

- It is equivalently a fiber sequence

$$\mathbf{B}A \to \mathbf{B}E \to \mathbf{B}G,$$

  or a principal $\mathbf{B}A$-bundle over $\mathbf{B}G$.
- Such group extensions are classified by maps $\mathbf{B}G \to \mathbf{B}\mathbf{B}A$, or elements of the “second group cohomology” $H^2(G, A)$.

Galois theory

- An object A is called a *Galois object* if A is an $\mathrm{Aut}(A)$-torsor.